

THE N -QUEENS PROBLEM ON A SYMMETRIC TOEPLITZ MATRIX

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ABSTRACT. We consider the problem of placing n nonattacking queens on a symmetric $n \times n$ Toeplitz matrix. As in the N -queens Problem on a chessboard, two queens may attack each other if they share a row or a column in the matrix. However, the usual diagonal restriction is replaced by specifying that queens may attack other queens that occupy squares with the same number value in the matrix. We will show that n nonattacking queens can be placed on such a matrix if and only if $n \equiv 0, 1 \pmod{4}$.

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The N -queens Problem, a generalization of the original 8-queens problem, asks whether n nonattacking queens can be placed on an $n \times n$ chessboard in such a way that no queen can attack another, i.e., so that no two queens are placed in the same row or column or on the same diagonal. The problem has been extensively studied since the mid-1800s; for a brief summary of the history, see [1, 3, 5]. A familiar extension of the problem (due to Polya) asks the same question for queens placed on a toroidal chessboard. Vardi also considers the toroidal semiqueens problem, in which a semiqueen may move “...like a rook or bishop, but only on positive broken diagonals.” [4] (See Figure 1.) To clarify the phrase “positive broken diagonals” without reference to a figure, number the rows and columns of an $n \times n$ chessboard from 1 to n , starting with the top left square. The k th positive broken diagonal then consists of all squares labeled (i, j) for $i + j = k + 1 \pmod{n}$.

In this paper we modify the chessboard again; now it corresponds to a symmetric Toeplitz matrix. A Toeplitz matrix has constant negative diagonals, i.e., entry (i_1, j_1) equals entry (i_2, j_2) whenever $i_1 - j_1 = i_2 - j_2$ [2]. In a symmetric Toeplitz matrix, we further require that entry (i_1, j_1) equal entry (i_2, j_2) whenever $|i_1 - j_1| = |i_2 - j_2|$. Queens may attack each other if they share a row or a column, or if both are on squares belonging to a set of the form $D_k = \{(i, j) \mid |i - j| = k\}$. We then ask when n nonattacking queens can be placed on such an $n \times n$ chessboard. We will show that n nonattacking queens can be placed on such a chessboard if and only if $n \equiv 0, 1 \pmod{4}$.

To solve this problem we will consider the matrix T_n with entries given by $t_{ij} = |i - j|$, for $i, j = 1, \dots, n$. We say that T_n is solvable if and only if we can select n entries from T_n with values $0, \dots, n-1$ so that no two entries selected lie in the same row or the same column. Equivalently, T_n is solvable if and only if there exists a permutation $f : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ such that $|f(i) - i|$ assumes each of the values $\{0, 1, \dots, n-1\}$ exactly once. (We will use both formulations throughout this paper.) A solution set for T_n is a set of pairs $S_n = \{(i, f(i)) \mid i = 1, \dots, n\}$ where f is a permutation demonstrating T_n 's solvability. For example, as indicated in Figure

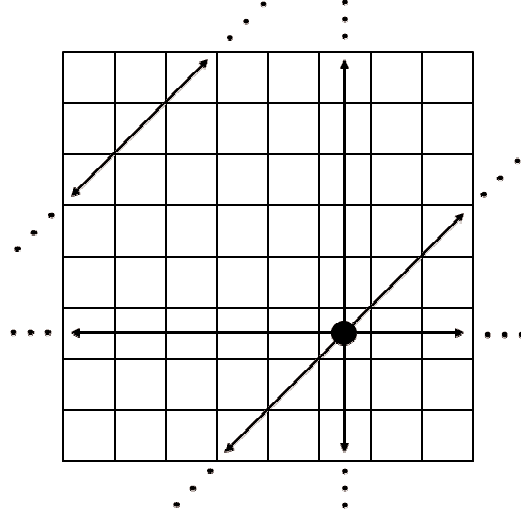


FIGURE 1.

A	B	C	D	E	F	G	H
B	A	B	C	D	E	F	G
C	B	A	B	C	D	E	F
D	C	B	A	B	C	D	E
E	D	C	B	A	B	C	D
F	E	D	C	B	A	B	C
G	F	E	D	C	B	A	B
H	G	F	E	D	C	B	A

1a) Placing a queen on the circle threatens queens on lines.

0	1	2	3
1	0	1	2
2	1	0	1
3	2	1	0

1b) T_4 is solvable.

0	1	2	3	4
1	0	1	2	3
2	1	0	1	2
3	2	1	0	1
4	3	2	1	0

1c) T_5 is solvable.

FIGURE 2.

2, solution sets for T_4 and T_5 are, respectively, $S_4 = \{(1, 3), (2, 2), (3, 4), (4, 1)\}$ and $S_5 = \{(1, 4), (2, 2), (3, 5), (4, 3), (5, 1)\}$.

Theorem 1. T_n is solvable if and only if $n \equiv 0, 1 \pmod{4}$.

Proof. We use a counting argument to show that T_n is not solvable for $n \equiv 2, 3 \pmod{4}$, then construct a solution set for T_n when $n \equiv 0, 1 \pmod{4}$.

Suppose $f : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ is a solution for T_n . As f is a permutation, it follows that

$$(1) \quad \sum_{i=1}^n (f(i))^2 = \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}.$$

As $|f(i) - i|$ achieves each of the values $\{0, \dots, n-1\}$ exactly once, we have

$$(2) \quad \sum_{i=1}^n |f(i) - i|^2 = \sum_{j=0}^{n-1} j^2 = \frac{(n-1)n(2n-1)}{6}.$$

However, it is also the case that

$$\sum_{i=1}^n |f(i) - i|^2 = \sum_{i=1}^n (f(i) - i)^2 = \sum_{i=1}^n (f(i))^2 - 2 \sum_{i=1}^n i f(i) + \sum_{i=1}^n i^2.$$

Simplifying, we obtain

$$\sum_{i=1}^n |f(i) - i|^2 = 2 \sum_{i=1}^n i^2 - 2 \sum_{i=1}^n i f(i).$$

Using (1) and (2) gives

$$\frac{(n-1)n(2n-1)}{6} = 2 \left(\frac{n(n+1)(2n+1)}{6} \right) - 2 \sum_{i=1}^n i f(i)$$

or

$$\sum_{i=1}^n i f(i) = \frac{1}{12} (2n(n+1)(2n+1) - (n-1)n(2n-1)).$$

Simplifying the right-hand side and recognizing that the left-hand side is an integer yields

$$n(2n^2 + 9n + 1) \equiv 0 \pmod{12}.$$

Suppose first that $n \equiv 2 \pmod{4}$. Since n is even, $2n^2 + 9n + 1$ is odd. But this together with $n(2n^2 + 9n + 1) \equiv 0 \pmod{12}$ implies $n \equiv 0 \pmod{4}$, a contradiction. Thus no solutions exist for $n \equiv 2 \pmod{4}$. Now assume that $n \equiv 3 \pmod{4}$. A short calculation shows that $2n^2 + 9n + 1 \equiv 2 \pmod{4}$, so $n(2n^2 + 9n + 1) \equiv 2n \equiv 2 \pmod{4}$ (under the assumption that $n \equiv 3 \pmod{4}$). But $n(2n^2 + 9n + 1) \equiv 2 \pmod{4}$ contradicts $n(2n^2 + 9n + 1) \equiv 0 \pmod{12}$. Therefore there are no solutions when $n \equiv 3 \pmod{4}$.

It remains to show that T_n is solvable when $n \equiv 0, 1 \pmod{4}$. The proof will be done by induction on n .

We begin by defining two matrices that can be obtained from T_n .

- T_n^* is the matrix obtained from T_n by deleting the first column and the last row of T_n .
- T_n^{**} is obtained from T_n by deleting the first and last row as well as the first and the $(n-1)^{st}$ column of T_n .

Figure 3 illustrates T_n^* and T_n^{**} . We will call T_n^* solvable if we can select entries with values $0, \dots, n-2$, one from each row and each column; likewise T_n^{**} will be called solvable if we can select entries with values $0, \dots, n-3$, one from each row and each column. Note that if T_n^* is solvable, then so is its transpose, $(T_n^*)^T$, which is obtained from T_n by removing the first row and the last column. We will not distinguish between T_n^* and $(T_n^*)^T$. The same holds for T_n^{**} and $(T_n^{**})^T$.

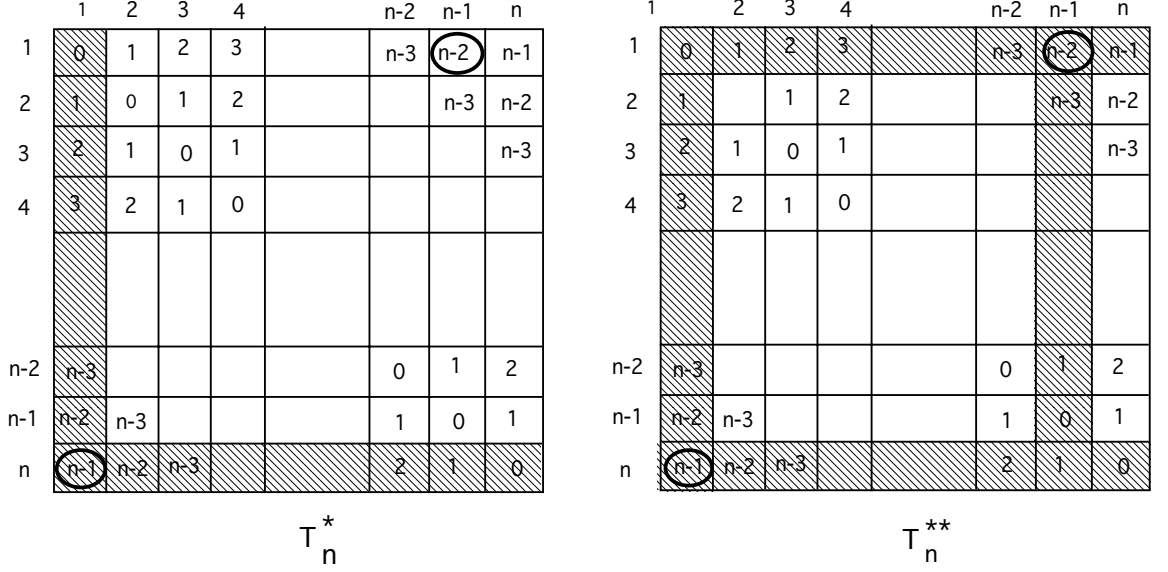


FIGURE 3.

If T_n is solvable, without loss of generality we can assume that the $(n, 1)$ entry was selected as the entry of value $n - 1$, see Figure 3. By symmetry we may assume that the $(1, n - 1)$ entry was selected as the entry of value $n - 2$. It is easy to see that removing the $(n, 1)$ entry from the solution set of T_n gives a solution set for T_n^* and removing both the $(n, 1)$ and the $(1, n - 1)$ entries gives a solution for T_n^{**} . Similarly adding one or both of these entries to the solution of T_n^{**} results in solutions for T_n^* and T_n respectively. We conclude that, for fixed n , either all three of T_n, T_n^* and T_n^{**} are solvable or none of them is.

Solutions for T_4 and T_5 are provided in Figure 2. We assume the result for all $k < n$. The solution set depends on the remainder of $n \bmod 3$, so we write $n = 3r + s$, where s may be 0, 1, or 2. Cases 2 and 3 are similar to Case 1 and thus will be written out in less detail.

Case 1: $n = 3r + 0$. We build a solution set S_n as follows: Select entries $(n, 1), (n - 1, 2), \dots, (n - r + 1, r) = (2r + 1, r)$ which have values $n - 1, n - 3, \dots, n - 2r + 1 = r + 1$ respectively. Also select entries $(1, n - 1), (2, n - 2), \dots, (r - 1, n - (r - 1)) = (r - 1, 2r + 1)$ which have values $n - 2, n - 4, \dots, n - 2r + 2 = r + 2$ and finally select the entry $(n - r, n) = (2r, n)$ with value r . Thus far S_n contains all values $r, r + 1, \dots, n - 1$ and no two selections are in the same row or column.

The entries that are still available to be selected (i.e., those lying in a row or column not yet selected) have indices (a, b) with $r \leq a \leq 2r - 1$ and $r + 1 \leq b \leq 2r$. As each entry is equal to the difference between its row and column index, we can subtract $r - 1$ from each of the row and column indices and obtain the matrix with row labels 1 through r and column labels 2 through $r + 1$. The matrix we obtain is T_{r+1}^* with the last row and the first column deleted, i.e., it is T_{r+1}^* . Thus if T_{r+1}^* has a solution set S_{r+1}^* , adding this solution set to the entries we

already selected completes S_n . By induction we know T_{r+1}^* is solvable as long as $r+1 \equiv 0$ or $1 \pmod{4}$. However it is easy to check that $n \equiv 0$ or $1 \pmod{4}$ if and only if $r+1 \equiv 0$ or $1 \pmod{4}$.

Case 2: $n = 3r + 1$. Select entries $(n, 1), (n-1, 2), \dots, (n-r+1, r) = (2r+2, r)$ and $(1, n-1), (2, n-2), \dots, (r, n-r) = (r, 2r+1)$ and $(n-r, n) = (2r+1, n)$. So far we have selected values with entries $r, r+1, \dots, n-1$. The entries that are still available to be selected have indices (a, b) such that $r+1 \leq a \leq 2r$ and $r+1 \leq b \leq 2r$. By subtracting r from both the column and row labels, we see that the matrix that remains is T_r . Again, it is easy to check that $n \equiv 0$ or $1 \pmod{4}$ if and only if $r \equiv 0$ or $1 \pmod{4}$. Thus, by the induction hypothesis, we can complete the solution set S_n .

Case 3: $n = 3r + 2$. Begin by selecting entries $(n, 1), (n-1, 2), \dots, (n-r+1, r) = (2r+3, r)$ and $(1, n-1), (2, n-2), \dots, (r, n-r) = (r, 2r+2)$ and $(n-(r+1), n) = (2r+1, n)$. The values we have selected so far are $r+1, \dots, n-1$. The entries that are still available to be selected are those with indices (a, b) such that $r+1 \leq a \leq 2r$ or $a = 2r+2$ and $r+1 \leq b \leq 2r+1$. By subtracting $r-1$ from all row and column labels we see that these are the entries of the transpose of T_{r+3}^{**} which by the induction hypothesis is solvable as long as $r+3 \equiv 0$ or $1 \pmod{4}$. Again we see that $n \equiv 0$ or $1 \pmod{4}$ if and only if $r+3 \equiv 0$ or $1 \pmod{4}$, and thus we can complete the solution set S_n . \square

A symmetric Toeplitz chessboard is an $n \times n$ chessboard in which the squares are labeled with numbers so that the labels along each descending diagonal are constant and the labeling is symmetric with respect to the main descending diagonal. A queen placed on the chessboard may attack other queens in the same row or column as well as those placed on a square with the same number value.

Corollary 2. *It is possible to place n queens on an $n \times n$ symmetric Toeplitz chessboard so that no two queens may attack each other if and only if $n \equiv 0, 1 \pmod{4}$.*

Proof. Without loss of generality assume the squares of the chessboard are labeled with the entries of T_n . Then placing the nonattacking queens on the chessboard is equivalent to finding a solution for T_n . The result follows by Theorem 1. \square

Corollary 2 establishes the existence (and nonexistence) of solutions to the N -queens Problem on the particular modification of the chessboard discussed in this paper. However, variants of this problem analogous to those on other chessboards remain to be addressed. For instance:

- (1) What is the minimum number of queens necessary for each square of the board to either contain a queen or to be attacked by at least one queen? (What is the minimum cardinality of a dominating set, i.e., what is the domination number?)
- (2) When $n \equiv 2, 3 \pmod{4}$, what is the maximum number of queens that can be placed on the board so that no queen can attack another? (What is the maximum cardinality of an independent dominating set?)
- (3) Substitute another type of chess piece for queens and ask analogous questions.
- (4) How many (fundamental) solutions exist for each question?

	1	2	3	4		$n-2$	$n-1$	n
1	0	1	2	3		$n-3$	$n-2$	$n-1$
2	1	0	1	2			$n-3$	$n-2$
3	2	1	0	1				$n-3$
4	3	2	1	0			$n-5$	
						$n-7$		
$n-2$	$n-3$			$n-6$		0	1	2
$n-1$	$n-2$	$n-3$	$n-4$			1	0	1
n	$n-1$	$n-2$	$n-3$			2	1	0

Theorem 3 resolves the second question.

Proof. We provide such a placement of $n - 1$ nonattacking queens as follows.

- (1) Place a queen on square $(1, 1)$.
- (2) For each i from 1 to $\lfloor \frac{n}{2} \rfloor - 1$, place a queen on square $(n + 1 - i, i + 1)$.
- (3) For each j from 3 to $\lfloor \frac{n}{2} \rfloor + 1$, place a queen on square $(j, n + 3 - j)$.

Case 1: n is even. Examine the rows occupied by queens. Step 1 places a queen in row 1; step 2 places queens in rows $\frac{n}{2} + 2, \dots, n$; step 3 places queens in rows $3, \dots, \frac{n}{2} + 1$. The $n - 1$ queens occur once in every row with the exception of row 2 only. Similarly, we have queens in column 1 (step 1), in columns $2, \dots, \frac{n}{2}$ (step 2), and in columns $\frac{n}{2} + 2, \dots, n$ (step 3). Every column except column $\frac{n}{2} + 1$ has exactly one queen. Calculating the differences of the row and column indices shows that there is exactly one queen in each of the diagonals indexed by 0 (step 1), by $2, 4, \dots, n - 2$ (step 2), and by $1, 3, \dots, n - 3$ (step 3). (The diagonal indexed by $n - 1$ has no queen.)

Case 2: n is odd. Rows occupied by a queen are row 1 (step 1), rows $\frac{n+3}{2}, \dots, n$ (step 2), and rows $3, \dots, \frac{n+1}{2}$ (step 3). Only row 2 has no queen, so the other rows have one queen each. Columns occupied by a queen are column 1 (step 1), columns $2, \dots, \frac{n+1}{2}$ (step 2), and columns $\frac{n+5}{2}, \dots, n$ (step 3). The only column with no queen is column $\frac{n+3}{2}$, so again we conclude that all the remaining columns contain exactly one queen. (Note that, as n is odd, $\frac{n+3}{2}$ may be written as $\lceil \frac{n}{2} \rceil + 1$, and $\lceil \frac{n}{2} \rceil + 1 = \frac{n}{2} + 1$ when n is even, so the expression $\lceil \frac{n}{2} \rceil + 1$ identifies the column without a queen for all values of n .) Finally, we check the diagonals occupied by queens: diagonal 0 (step 1), diagonals $1, 3, \dots, n-2$ (step 2), and diagonals $2, 4, \dots, n-3$ (step 3). As before, only the diagonal indexed by $n-1$ has no queen; the others have exactly one each. \square

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